Lecture 4
Math Prerequisite 1: Projective Geometry

Lin ZHANG, PhD
School of Software Engineering
Tongji University
Fall 2024

Outline

- Vector Operations
- Fundamentals of Projective Geometry

Vector operations

Vector representation

$$\vec{a} = x\vec{i} + y\vec{j} + z\vec{k} = \{x, y, z\}$$

Length (or norm) of a vector

$$|\vec{a}| = \sqrt{x^2 + y^2 + z^2}$$

Normalized vector (unit vector)

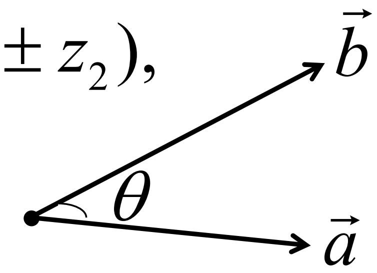
$$\frac{\vec{a}}{|\vec{a}|} = \{\frac{x}{|\vec{a}|}, \frac{y}{|\vec{a}|}, \frac{z}{|\vec{a}|}\}$$

We say $\vec{a} = \mathbf{0}$, if and only if $x = 0, y = 0, z = 0$

Vector operations

if $\vec{a} = (x_1, y_1, z_1),\ \vec{b} = (x_2, y_2, z_2),$

then $\vec{a} \pm \vec{b} = (x_1 \pm x_2, y_1 \pm y_2, z_1 \pm z_2),$



Dot product (inner product)

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos\theta = x_1x_2 + y_1y_2 + z_1z_2$$

Laws of dot product:

$$\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}, \vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$$

Theorem

$$\vec{a} \cdot \vec{b} = 0 \Leftrightarrow \vec{a} \perp \vec{b}$$ (why?)

Vector operations

Cross product

$$\vec{a}\times\vec{b}=\begin{vmatrix} i & j & k \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix}=\begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix} i+\begin{vmatrix} z_1 & x_1 \\ z_2 & x_2 \end{vmatrix} j+\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} k$$

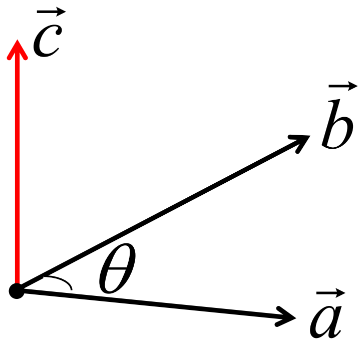
Vector operations

Cross product

$\vec{c} = \vec{a} \times \vec{b}$ is also a vector, whose direction is determined by the right-hand law and

$$\vec{c} \perp \vec{a}, \vec{c} \perp \vec{b}$$

$$|\vec{c}| = |\vec{a}||\vec{b}|\sin\theta$$



$\vec{c}$ represents the oriented area of the parallelogram taking $\vec{a}$ and $\vec{b}$ as two sides (easy to prove)

$$\overrightarrow{r_1} \times \overrightarrow{r_2} = -\overrightarrow{r_2} \times \overrightarrow{r_1}$$ (why?)

Vector operations

Cross product

Theorem

$$\vec{a} \parallel \vec{b} \Leftrightarrow \vec{a} \times \vec{b} = \mathbf{0}$$ (why?)

Theorem

$\vec{a} \parallel \vec{b} \Leftrightarrow \exists \lambda, \mu,$ they are not equal to zero at the same time, and $\lambda\vec{a} + \mu\vec{b} = \mathbf{0}$ (easy to understand)

Property

$$\overrightarrow{r_1} \times (\overrightarrow{r_2} + \overrightarrow{r_3}) = \overrightarrow{r_1} \times \overrightarrow{r_2} + \overrightarrow{r_1} \times \overrightarrow{r_3}$$

Vector operations

Cross product

Definition

Suppose $\vec{a} = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$ $\qquad \vec{a}^{\wedge} \triangleq \begin{bmatrix} 0 & -z_1 & y_1 \\ z_1 & 0 & -x_1 \\ -y_1 & x_1 & 0 \end{bmatrix}$

Then,

$$\vec{a} \times \vec{b} = \vec{a}^{\wedge} \vec{b}$$

Vector operations

Mixed product (scalar triple product or box product)

$$(\mathbf{a},\mathbf{b},\mathbf{c}) = (\mathbf{a}\times\mathbf{b})\cdot\mathbf{c} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Geometric Interpretation: it is the (signed) volume of the parallelepiped defined by the three vectors given

Vector operations

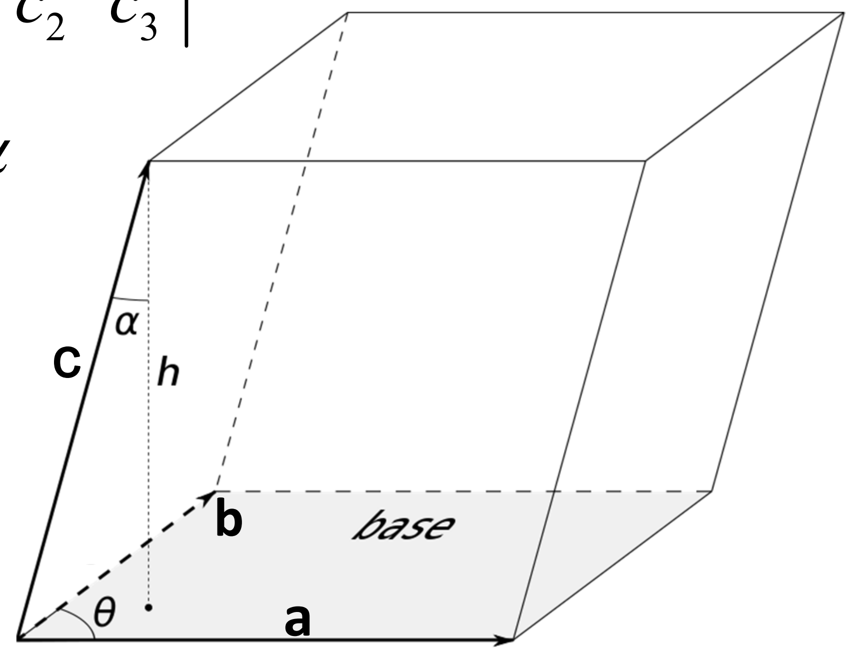
Mixed product (scalar triple product or box product)

$$(\mathbf{a},\mathbf{b},\mathbf{c}) = (\mathbf{a}\times\mathbf{b})\cdot\mathbf{c} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c} = |\mathbf{a}\times\mathbf{b}||\mathbf{c}|\cos\alpha$$

$$= |\mathbf{a}||\mathbf{b}|\sin\theta \cdot |\mathbf{c}|\cos\alpha$$

Base $\qquad$ *h*



Vector operations

Mixed product (scalar triple product or box product)

$$(\mathbf{a},\mathbf{b},\mathbf{c}) = (\mathbf{a}\times\mathbf{b})\cdot\mathbf{c} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Property:

$$(\mathbf{a},\mathbf{b},\mathbf{c}) = (\mathbf{b},\mathbf{c},\mathbf{a}) = (\mathbf{c},\mathbf{a},\mathbf{b})$$

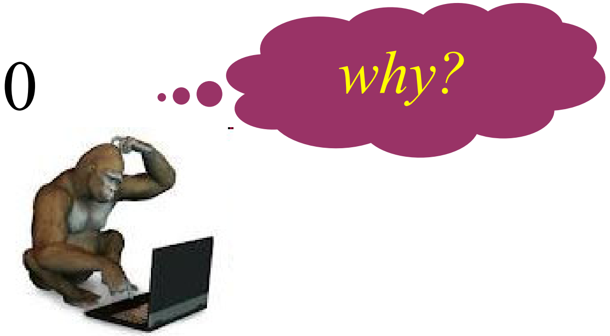
$$(\mathbf{a},\mathbf{b},\mathbf{c}) = -(\mathbf{b},\mathbf{a},\mathbf{c}) = -(\mathbf{a},\mathbf{c},\mathbf{b})$$

Vector operations

Mixed product (scalar triple product or box product)

Theorem

$\mathbf{a}, \mathbf{b}, \mathbf{c}$ are coplanar $\Leftrightarrow (\mathbf{a}, \mathbf{b}, \mathbf{c}) = 0$



$\mathbf{a}, \mathbf{b}, \mathbf{c}$ are coplanar $\Leftrightarrow \exists \lambda, \mu, v,$ they are not equal to zero at the same time, and $\lambda \mathbf{a} + \mu \mathbf{b} + v \mathbf{c} = \mathbf{0}$

Outline

- Vector Operations
- Fundamentals of Projective Geometry

Foundations of Projective Geometry

- What is homogeneous coordinate?

  For a **normal** point $(x, y)^T$ on a plane π_0,

  its homogenous coordinate is $k(x, y, 1)^T$, where k can be **any** non-zero real number

  Homogenous coordinate for a point is not unique

  For a homogenous coordinate (normal point) $(x', y', z')^T$

  we can rewrite it as $(x'/z', y'/z', 1)^T$

  normalized homogenous coordinate

Foundations of Projective Geometry

- What is homogeneous coordinate?

For a **normal** point $(x, y)^T$ on a plane π_0 ,
its homogenous coordinate is $k(x, y, 1)^T$, where k can be **<u>any</u>** non-zero real number

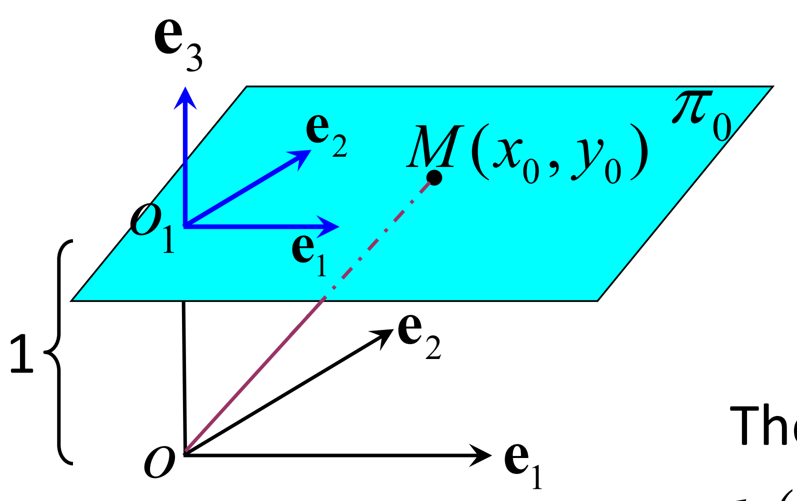
Converting from homogenous coordinate (normal point) to inhomogeneous coordinate,

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} \Rightarrow \begin{pmatrix} \frac{x'}{z'} \\ \frac{y'}{z'} \end{pmatrix}$$

Foundations of Projective Geometry

- What is homogeneous coordinate?

Geometric interpretation



In plane π_0, in the 2D frame $(o_1 : \mathbf{e}_1, \mathbf{e}_2)$, one point $M : (x_0, y_0)$

Coordinate of any point (except O) on line OM in the frame $(o : \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ is the homogeneous coordinate of M

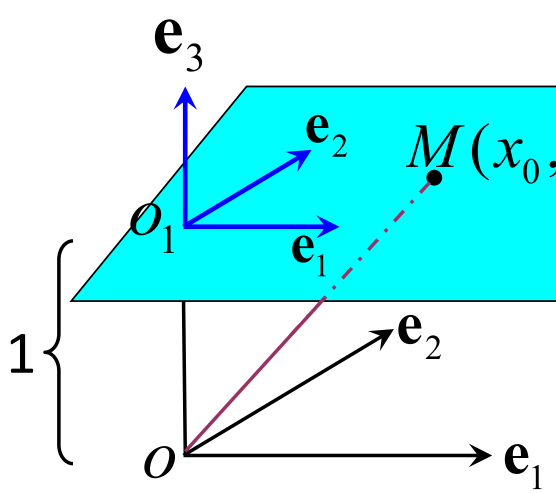
These points can be represented as

$$k\left(x_0, y_0, 1\right)^T, k \neq 0$$

Foundations of Projective Geometry

- What is homogeneous coordinate?

Geometric interpretation



In plane π_0, in the 2D frame $(o_1 : \mathbf{e}_1, \mathbf{e}_2)$, one point $M : (x_0, y_0)$

Coordinate of any point (except O) on line OM in the frame $(o : \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ is the homogeneous coordinate of M

These points can be represented as

$$k(x_0, y_0, 1)^T, k \neq 0$$

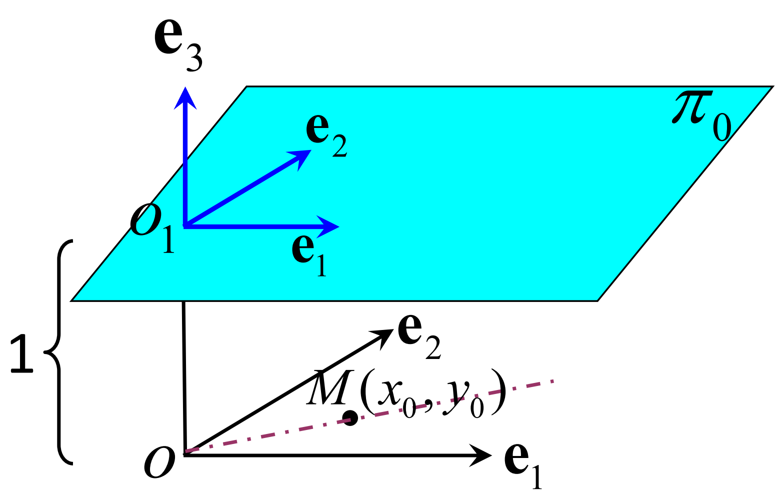
How about a line passing through O and parallel to π_0*?*

Foundations of Projective Geometry

- What is homogeneous coordinate?

Geometric interpretation

How about a line passing through O and parallel to π_0?



Consider a line passing through O and $M(x_0, y_0, 0)^T$
We define: it meets π_0 at an infinity point, and also the homogeneous coordinate of such a point can be represented as points on OM

So, the infinity point has the form $(kx_0, ky_0, 0)^T$

Foundations of Projective Geometry

- What is homogeneous coordinate?

Normal case:

line $k(x_0, y_0, 1)$ $(k!=0)$

↕

a normal point (x_0, y_0) on the plane π_0

↕

The homogeneous coordinate of this normal point is $k(x_0, y_0, 1)$

make an analogy

abnormal case:

line $(kx_0, ky_0, 0)$ $(k!=0)$

↕

Define: it meets π_0 at an **infinity point**

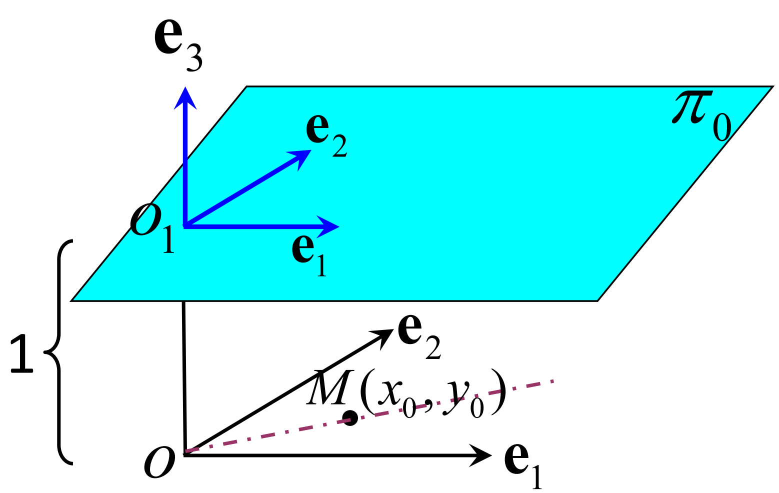
↕

The homogeneous coordinate of this infinity point is $k(x_0, y_0, 0)$

Foundations of Projective Geometry

- What is homogeneous coordinate?

Geometric interpretation



How about a line passing through O and parallel to π_0?

One infinity point determines an orientation

We define: all infinity points on π_0 comprise an **infinity line**

In fact, plane $o\mathbf{e}_1\mathbf{e}_2$ meets π_0 at the infinity line

Foundations of Projective Geometry

π_0 + infinity line = Projective plane

An Euclidean plane

In fact, any Euclidean space $\mathbb{R}^n$ can be extended to a projective space by

Properties of a projective plane

- Two points determine a line; two lines determine a point (the second claim is not correct in the normal Euclidean plane)
- Two parallel lines intersect at an infinity point; that means one infinity point corresponds to a specific orientation
- Two parallel planes intersect at the infinity line

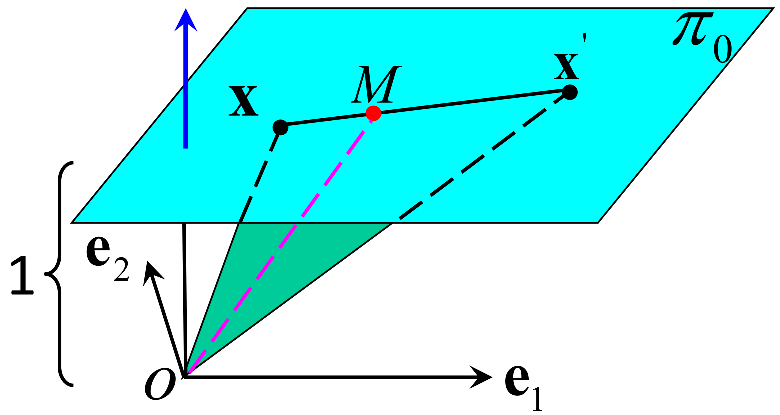
Properties of a 3D projective space

- All properties on 2D projective plane can be kept
- More on infinities
  - On 2-D projective plane, all infinity points form an infinity line; in 3-D projective space, all infinity points form an infinity plane; or in other words, all infinity lines form an infinity plane

Foundations of Projective Geometry

- Lines in the homogeneous coordinate

On a projective plane, please determine the line passing two points $\mathbf{x} = (x_1, y_1, z_1)^T, \mathbf{x}' = (x_2, y_2, z_2)^T$



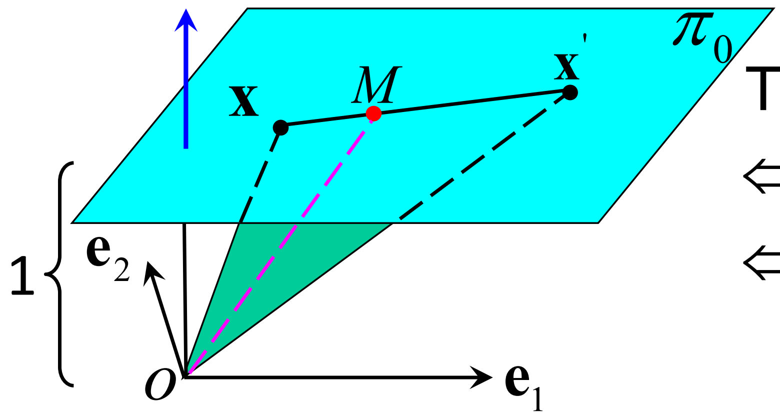
$o\mathbf{x}, o\mathbf{x}'$ determine two lines

$\mathbf{xx}'$ actually is the intersection between $o\mathbf{xx}'$ and π_0

Foundations of Projective Geometry

- Lines in the homogeneous coordinate

On a projective plane, please determine the line passing two points $\mathbf{x} = (x_1, y_1, z_1)^T, \mathbf{x}' = (x_2, y_2, z_2)^T$



Thus, $M(x, y, z)$ locates on $\mathbf{xx}'$

$\Leftrightarrow$ oM resides on the plane $o\mathbf{xx}'$

$\Leftrightarrow$ $o\mathbf{x}, oM, o\mathbf{x}'$ are coplanar

$$\Leftrightarrow \begin{vmatrix} x & y & z \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} = 0$$

Foundations of Projective Geometry

- Lines in the homogeneous coordinate

On a projective plane, please determine the line passing two points $\mathbf{x} = (x_1, y_1, z_1)^T, \mathbf{x}' = (x_2, y_2, z_2)^T$

$$\begin{vmatrix} x & y & z \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} = 0 \Leftrightarrow \begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix} x + \begin{vmatrix} z_1 & x_1 \\ z_2 & x_2 \end{vmatrix} y + \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} z = 0$$

$$\left(\begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix}, \begin{vmatrix} z_1 & x_1 \\ z_2 & x_2 \end{vmatrix}, \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} \right)^T$$

Homogeneous coordinate of the line

Homogeneous coordinate of the infinity line is $(0,0,1)^T$

Foundations of Projective Geometry

- Lines in the homogeneous coordinate

On a projective plane, please determine the line passing two points $\mathbf{x}=(x_1,y_1,z_1)^T, \mathbf{x}^{'}=(x_2,y_2,z_2)^T$

$$\begin{vmatrix} x & y & z \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} = 0 \Leftrightarrow \begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix} x + \begin{vmatrix} z_1 & x_1 \\ z_2 & x_2 \end{vmatrix} y + \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} z = 0$$

$$\left(\begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix}, \begin{vmatrix} z_1 & x_1 \\ z_2 & x_2 \end{vmatrix}, \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} \right)^T$$

Theorem

On the projective plane, the line passing two points $\mathbf{x}, \mathbf{x}^{'}$ is

$\mathbf{l} = \mathbf{x} \times \mathbf{x}^{'}$

Foundations of Projective Geometry

- Lines in the homogeneous coordinate

A point $\mathbf{x} = (x_0, y_0, z_0)^T$ is on the line $\mathbf{l} = (a, b, c)^T$

$\Leftrightarrow$ $\mathbf{x}^T \mathbf{l} = 0$ (It is $\mathbf{x} \cdot \mathbf{l} = 0$)

Foundations of Projective Geometry

- Lines in the homogeneous coordinate

Theorem: On the projective plane, the intersection of two lines $\mathbf{l}, \mathbf{l}^{'}$ is the point $\mathbf{x} = \mathbf{l} \times \mathbf{l}^{'}$

Proof: Two lines $a_1 x + b_1 y + c_1 z = 0,\ a_2 x + b_2 y + c_2 z = 0$

$$\mathbf{l} = (a_1, b_1, c_1)^T, \mathbf{l}^{'} = (a_2, b_2, c_2)^T$$

Inhomogeneous form $\left(X = \frac{x}{z}, Y = \frac{y}{z} \right)$

$$\begin{cases} a_1 X + b_1 Y + c_1 = 0 \\ a_2 X + b_2 Y + c_2 = 0 \end{cases} \Rightarrow X = \frac{\begin{vmatrix} -c_1 & b_1 \\ -c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}, Y = \frac{\begin{vmatrix} a_1 & -c_1 \\ a_2 & -c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

Foundations of Projective Geometry

- Lines in the homogeneous coordinate

Theorem: On the projective plane, the intersection of two lines $\mathbf{l}, \mathbf{l}'$ is the point $\mathbf{x} = \mathbf{l} \times \mathbf{l}'$

Homogenous form of the cross point is

$$\mathbf{x} = k\left(\frac{\begin{vmatrix} -c_1 & b_1 \\ -c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}, \frac{\begin{vmatrix} a_1 & -c_1 \\ a_2 & -c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}, 1 \right) \xrightarrow{\text{let } k = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} \mathbf{x} = \left(\begin{vmatrix} -c_1 & b_1 \\ -c_2 & b_2 \end{vmatrix}, \begin{vmatrix} a_1 & -c_1 \\ a_2 & -c_2 \end{vmatrix}, \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \right)$$

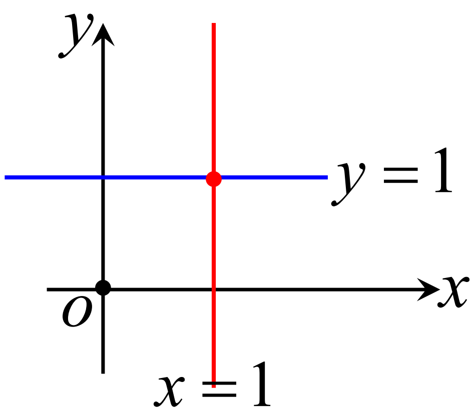
$$\Leftrightarrow \mathbf{x} = \left(\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}, \begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}, \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \right) \longrightarrow \mathbf{x} = \mathbf{l} \times \mathbf{l}'$$

Foundations of Projective Geometry

- Lines in the homogeneous coordinate

Theorem: On the projective plane, the intersection of two lines $\mathbf{l}, \mathbf{l}'$ is the point $\mathbf{x} = \mathbf{l} \times \mathbf{l}'$

Example: find the cross point of the lines $x = 1, y = 1$



Homogeneous form

$$\begin{cases} x_1 + 0x_2 + (-1)x_3 = 0 \\ 0x_1 + 1x_2 + (-1)x_3 = 0 \end{cases}$$

Homogeneous coordinates of the two lines are $(1,0,-1)^T, (0,1,-1)^T$

Cross point is

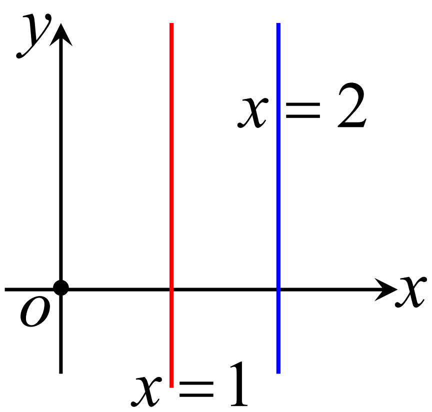
$$(1,0,-1)^T \times (0,1,-1)^T = (1,1,1)$$

Foundations of Projective Geometry

- Lines in the homogeneous coordinate

Theorem: On the projective plane, the intersection of two lines $\mathbf{l}, \mathbf{l}'$ is the point $\mathbf{x} = \mathbf{l} \times \mathbf{l}'$

Example: find the cross point of the lines $x = 1, x = 2$



Homogeneous form

$$\begin{cases} 1x_1 + 0x_2 + (-1)x_3 = 0 \\ 1x_1 + 0x_2 + (-2)x_3 = 0 \end{cases}$$

Homogeneous coordinates of the two lines are $(1,0,-1)^T, (1,0,-2)^T$

Cross point is

$$(1,0,-1)^T \times (1,0,-2)^T = \begin{vmatrix} i & j & k \\ 1 & 0 & -1 \\ 1 & 0 & -2 \end{vmatrix} = (0,1,0)$$

Foundations of Projective Geometry

- Duality

In projective geometry, lines and points can swap their positions

$$\mathbf{x}^T\mathbf{l} = 0 \quad \textit{How to interpret?}$$

If $\mathbf{x}$ is a variable, it represents the points lying on the line $\mathbf{l}$;

If $\mathbf{l}$ is a variable, it represents the lines passing a fixed point $\mathbf{x}$

The line passing two points $\mathbf{x}, \mathbf{x}^{'}$ is $\mathbf{l} = \mathbf{x} \times \mathbf{x}^{'}$

The cross point of two lines $\mathbf{l}, \mathbf{l}^{'}$ is $\mathbf{x} = \mathbf{l} \times \mathbf{l}^{'}$

Duality Principle: To any theorem of projective geometry, there corresponds a dual theorem, which may be derived by interchanging the roles of points and lines in the original theorem

More results you need to be familiar

- A set of parallel lines intersect at the same infinity point
- The homogeneous coordinate of the infinity line is $k(0,0,1)$
- The infinity point of a line can be identified as its intersection with the infinity line. E.g, on a projective plane, the infinity point of the X-axis is $k(1,0,0)$

  (Note: since the infinity point actually represents a direction, usually it is represented as a norm vector, for example (1, 0, 0)

- In 3D projective space, the infinity plane π_∞ are composed of points of the form $(x_1, x_2, x_3, x_4 = 0)$; You can also consider that the infinity plane comprises all the possible directions in 3D space

More results you need to be familiar

- Projective transformation

π_0, π_1 are two projective planes, $\mathbf{H} \in \mathbb{R}^{3\times3}$ is a matrix

$$\forall \mathbf{x}' \in \pi_1, \exists \text{ a unique } \mathbf{x} \in \pi_0, \mathbf{x}' = \mathbf{H}\mathbf{x}$$

$$\forall \mathbf{x} \in \pi_0, \exists \text{ a unique } \mathbf{x}' \in \pi_1, \mathbf{x} = \mathbf{H}^{-1}\mathbf{x}'$$

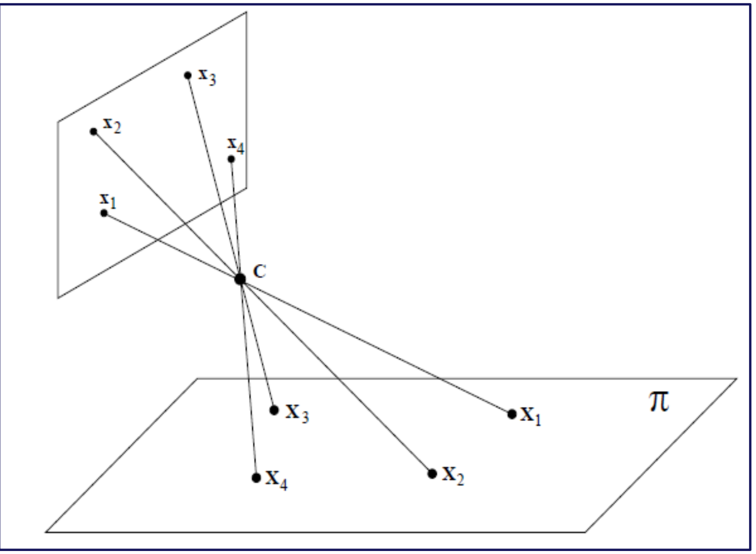
We say π_0, π_1 can be projectively transformed to each other and $\mathbf{H}$ is the projective transformation matrix between them. For the 2-D case, $\mathbf{H}$ is also called as homography

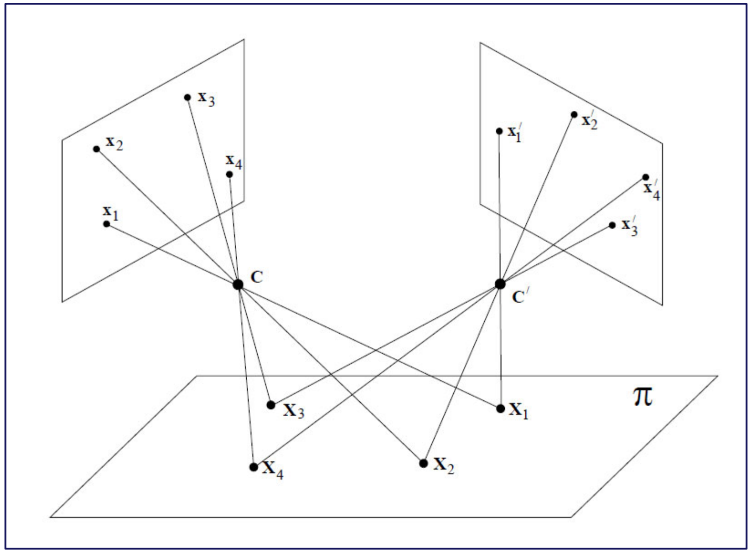
Note 1: If π_0 can be projectively transformed to π_1, the projective transformation from π_0 to π_1 is unique up to a scale factor

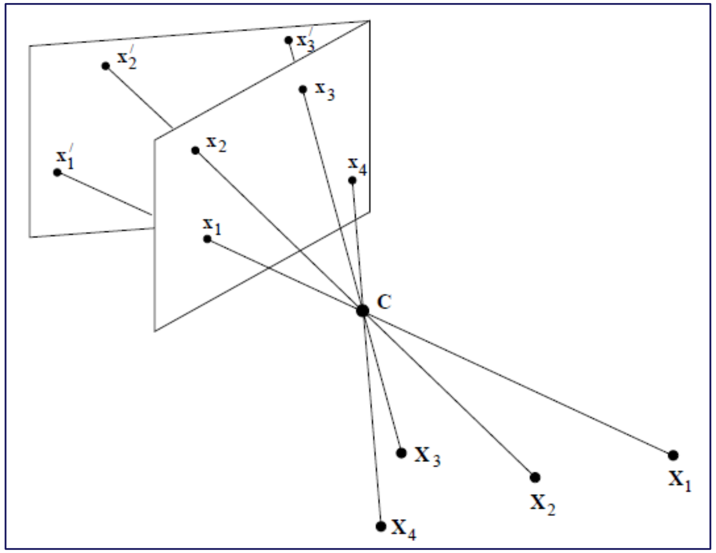
Note 2: The above definition is for 2D case. It can be straightforwardly extended to other dimensions

More results you need to be familiar

- Projective transformation (typical examples in CV)








Q+A