To prove:

If x^* is primal feasible,$(\alpha^*,\beta^*)$is dual feasible, and the duality gap is 0, i.e., $f_0(x^*)-g(\alpha^*,\beta^*)=0$, then x^* and (α^*,β^*) are optimal solutions for the primal and dual problems, respectively, and the primal problem has strong duality.

Proof:

1. Given:

1) x^* is primal feasible (satisfies primal constraints).

2) (α^*,β^*) is dual feasible (satisfies dual constraints).

3) The duality gap is 0: $f_0(x^*)-g(\alpha^*,\beta^*)=0$.

4) $g(\alpha,\beta)=\min_{x\in\mathcal{D}} l(x,\alpha,\beta)=\min_{x\in\mathcal{D}} f_0(x)+\sum_{i=1}^{m}\alpha_i f_i(x)+\sum_{i=1}^{p}\beta_i h_i(x)$

2. Implication of 0 Duality Gap:

The duality gap being 0 means that the objective value of the primal problem at x^* is equal to the objective value of the dual problem at (α^*,β^*) i.e., $f_0(x^*)=g(\alpha^*,\beta^*)$.

3. Optimality:

1) Since x^* is primal feasible and the primal objective is equal to the dual objective, x^* must be optimal. The reason is as below:

$f_0(x^*)=g(\alpha^*,\beta^*)\le f_0(x)$

2) Since x^* is primal feasible and the primal objective is equal to the dual objective, (α^*,β^*) must be optimal. The reason is as below:

$g(\alpha,\beta)\le f_0(x^*)=g(\alpha^*,\beta^*)$

4. Conclusion:

If the duality gap is 0, both x^* and (α^*,β^*) are optimal solutions, and the primal problem has strong duality (i.e., the primal and dual optimal values are equal).